Lecture 9 — Study of the Laplace operator $\Delta = \partial_1^2 + ... + \partial_n^2$

-This is the study of Harmonic functions :

$$\Delta u = 0 \quad in \quad \Omega$$

- Poisson eq :

$$\Delta u = f \quad in \quad \Omega$$

-Neumann bc :

$$\partial_v u = g \quad on \quad \partial\Omega$$

-Robin bc :

$$ku + \partial_v u = g \quad on \quad \partial\Omega$$

-Methods generalize to variable coefficients, higher order, elliptic linear and non-linear eq and systems.

Second Order Linear PDE

Consider L a 2^{nd} differential operator

$$Lu = \sum_{i,j}^{n} a_{ij}(x)\partial_i\partial_j u \qquad A = (a_{ij}) - symmetric. \tag{1}$$

The symmetry here is not imposed; once can always rewrite the PDE such that we have symmetry. Eg

$$Lu = a_{11}(x)\partial^2 u + \underbrace{a_{12}(x)\partial_1\partial_2 u + a_{21}(x)\partial_2\partial_1 u}_{rewrite} + a_{22}(x)\partial_2^2 u \tag{2}$$

$$\rightarrow \qquad (a_{12}(x) + a_{21}(x))\partial_1\partial_2 u \tag{3}$$

$$\implies \frac{a_{12}(x) + a_{21}(x)}{2}\partial_1\partial_2 u + \frac{a_{12}(x) + a_{21}(x)}{2}\partial_2\partial_1 u \tag{4}$$

to obtain symmetry. Suppose we want to introduce a change of coordinate from $x_{\mathbb{R}^n} \mapsto y_{\mathbb{R}^n}$.
Let $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a smooth mapping,

$$y_i = \phi_i(x). \tag{5}$$

We take derivatives by chain and product rule

$$\frac{\partial^2 u(\phi)}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\partial u}{\partial x_i}\right) \tag{6}$$

$$= \frac{\partial}{\partial x_j}\left(\frac{\partial u}{\partial y_k}\frac{\partial y_k}{\partial x_i}\right); \quad y_k = \phi_k(x) \tag{7}$$

$$= \frac{\partial^2 u}{\partial y_k \partial y_l}\frac{\partial \phi_k}{\partial x_i}\frac{\partial \phi_l}{\partial x_j} + \frac{\partial u}{\partial y_k}\frac{\partial^2 \phi_k}{\partial x_i \partial x_j} \tag{8}$$

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Highest orderer (principal) part becomes

$$L_{pr}u = \sum_{k,l} \frac{\partial^2 u}{\partial y_k \partial y_l} \underbrace{\left(\sum_{i,j} a_{ij}(x) \frac{\partial \phi_k}{\partial x_i} \frac{\phi_l}{\partial x_j} \right)}_{b_{kl}(x)} \tag{9}$$

Suppose $a_{ij}(x) = a_{ij}$, and $\phi(x) = Tx + c$, $\partial_i \phi_k = T_{ki}$. Note that ϕ defines a linear transformation and translation.

$$B = TAT^t.$$

One can choose T s.t

$$B = diag(\underbrace{1, ..., 1}_{n_+}, \underbrace{-1, ..., -1}_{n_-}, 0, ..., 0)$$

where $n_+ (respectively\ n_-)$ is numbers of positive eigenvalues of A.

$$n_+ = n\ or\ n_- = n :\ elliptic.$$

$$n_+ = n - 1,\ n_- = 1\ :\ hyperbolic.$$

$$n_+ = n - 1,\ n_- = 0\ :\ parabolic$$

The same can be done for variable coefficients case at each pt $x \in \Omega$.

Elliptic Case

Fig 9.1 if you want to simplify the eq. on an open set, then you have to solve,

$$a_{ij}(x) \partial_i \phi_k \partial_j \phi_l = \delta_{kl} \tag{10}$$

It can solved iff the Riemann-Christoffel tensor of A vanishes ie (A is flat) .

$$a_{ij}(x) \partial_i \phi_k \partial_j \phi_l = \psi(x) \delta_{kl} \quad \text{for some} \psi.$$

Now obstruction in $n = 2$. For $n \geq 3$, the obstruction is Cotton-Weyl tensor.

Structure of A

A constant, ϕlinear, $TAT^t = A$ generalized orthogonal transformation. (A):

$$for\ A = I \quad : \quad O(A) = O(n) = \{T \in R^{n \times n} : TT^t = I\}.$$

$$for\ A = diag(1, ...1, -1) \quad : \quad O(A) = O(n, 1)$$

$D\phi A D\phi^t = \psi A$ generalized conformal transformation. For $A = I\ :\ n = 2\ :$ very rich. For $n \geq 3$: only conformal transformations are combinations of translations, scaling, orthogonal and inversion. (Liouville's thm).

Fundamental Sol of Δ

Laplace : $\Delta\varphi = 0$ outside Ω. The solution for φ comes from the radial symmetry the Laplacian operator has, therefore setting $r = |x-y|$ and solving for $v := \psi(r)$ in $\Delta v = 0$

$$\implies \Delta v = \psi''(r) + \frac{n-1}{r}\psi'(r) = 0 \tag{11}$$

yielding

$$\psi(|x-y|) = \varphi(x) = C\int_\Omega \frac{f(y)}{|x-y|}dy.$$

Poisson (1813) : $\Delta\varphi = -4\pi C f$

$$E(x) = \begin{cases} \frac{1}{(2-n)S^{n-1}|x|^{n-1}} & n \geq 3 \\ \frac{ln|x|}{2\pi} & n = 2 \end{cases}$$

For $F \in C^1(\Omega) \cap C^0(\overline{\Omega})$,

$$\int_\Omega Div\ F\ dx = \int_{\partial\Omega} F\nu\ dS \qquad \nu \in T_x^\perp(\partial\Omega)\ unit\ outwards. \qquad (Gauss\ Div\ Theorem\ 1813)$$

Green's Identities (1828)

Let $u, v \in C^2(\Omega)\cap C^0(\overline{\Omega})$. Suppose $F = \vec{\bigtriangledown} u$ then $Div\ F = \Delta u$ and $F\nu = \partial_\nu u = \left(\frac{du}{dn} - \text{Fritz notation}\right)$, however Note that $\partial_\nu u$ signifies the directional derivative of u with respect to the *exterior* unit normal to $T(\partial\Omega)$ at $x \in \partial\Omega$; explicitly we have $F\nu = \sum_i \frac{\partial u}{\partial x_i}\nu_i$.
By the Gauss divergence theorem:

$$\int_\Omega \underbrace{\Delta u}_{Div\ F}\ dx = \int_{\partial\Omega} F\nu\ dS \qquad (Green\ \emptyset) \tag{12}$$

Suppose now that $F = u\vec{\bigtriangledown} v$. Then the $Div\ F = \vec{\bigtriangledown} u \cdot \vec{\bigtriangledown} v + u\Delta v\ and\ F\nu = u\partial_\nu v$, moreover

$$\int_\Omega \underbrace{\vec{\bigtriangledown} u \cdot \vec{\bigtriangledown} v + u\Delta v}_{Div\ F}\ dx = \int_{\partial\Omega} F\nu\ dS \qquad (Green\ I) \tag{13}$$

$$\int_\Omega u\Delta v - v\Delta u\ dx = \int_{\partial\Omega} u\partial_\nu v - v\partial_\nu u\ dS \qquad (Green\ II) \tag{14}$$

(where in the Fritz $\vec{\bigtriangledown} u \cdot \vec{\bigtriangledown} v$ was computed explicitly as $\sum_i u_{x_i}v_{x_i} = \sum_i \partial_i u\ \partial_i v$.)

Applications

a) In Green $\emptyset$,

$$If\ \Delta u = 0 \implies \int_{\partial\Omega} \partial_\nu u = 0.$$

$$If\ \partial_\nu u = 0 \implies \int_\Omega \Delta u = 0$$

b) Uniqueness theorem — In Green I, put $v = u$ with $\Delta u = 0$ then the "energy identity"

$$\int_\Omega |\vec{\nabla} u|^2 = \int_{\partial\Omega} u \partial_\nu u$$

If $u = 0$ *or* $\partial_\nu u = 0$ *on* $\partial\Omega$

$$\implies \int |\vec{\nabla} u|^2 = 0 \implies u \equiv const\ in\ \Omega, for\ u \in C^2(\overline{\Omega}).$$

c) In (Green I), $u = E$ Fig 9.2. $\Omega_\epsilon = B_R \backslash B_\epsilon$, *supp* $v \subset B_R$

$$\int_{\Omega_\epsilon} E\Delta v = \int_{\partial\Omega_\epsilon} E\partial_\nu v - v\partial_\nu E = -\int_{\partial B_\epsilon} E\partial_r v + \int_{\partial B_\epsilon} u\partial_r E$$

$$E\ \frac{1}{r^{n-2}}$$

$$\int^\epsilon \frac{r^{n-1}}{r^{n-2}} = \int^\epsilon r dr\ \epsilon^2$$

$$\partial_r R\ \frac{1}{r^{n-1}}$$